On Certain Class of Multivalent Analytic Functions Involving Linear Multiplier Operator and Subordination

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Abstract — In this paper, we introduced a new class of multivalent analytic functions in the open unit disk U defined in terms of a linear multiplier operator containing the generalized integral operator. The aim of the paper is to investigate inclusion relation, integral preserving property, and argument estimate by making use of subordination.

Index Terms —Multivalent function, integral operator, multiplier operator, subordination, generalized integral operator, inclusion relation and integral preserving property.

1. Introduction

Let A_p denote the class of function of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \text{ , (} p \in \mathbb{N} = 1,2,3 \dots), \quad (1.1)$$

that are analytic and p – valent in the open unit disk $U = \{z: z \in C, |z| < 1\}$.

Let $f, g \in A_{p,}$ where f is given by (1.1)and g is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}.$$

Then the Hadmard product (or convolution) $f * g$ of function f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z)$$

For two functions f and g analytic in$U$, we say that the function f is subordinate to g in U, denoted by $f \prec g$, if their exist a Schwarz function$w(z)$, which is analytic in U with

$$w(0) = 0 \text{ and } |w(z)| < |z| < 1, (z \in U),$$

such that

$$f(z) = g\big(w(z)\big), (z \in U).$$

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$and $f(U) \subset g(U)$.

For parameters$a, b \in C$ and $c \in C\backslash\mathbb{Z}_0^-$, $\mathbb{Z}_0^- = 0, -1, -2, \dots,$ the Gauss hyper geometric function

$${}_2F_1(a, b; c; z) = \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad (1.2)$$

where$(v)_k$ is the Pochhammer symbol defined by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 & : k = 0 \\ v(v+1) \dots (v+k-1) & : k \in \mathbb{N}. \end{cases} \quad (1.3)$$

The hyper geometric series in (1.2) converges absolutely for all$z \in U$, so that it represents an analytic function in U. We now introduce a function $f_{\mu,p}(a, b;\ c)(z)$ defined by

$$f_{\mu,p}(a, b;\ c)(z) = (1-\mu) z^p {}_2F_1(a, b; c; z) + \mu z [z^p {}_2F_1(a, b; c; z)]' \quad (1.4)$$

Where $z \in U$; $\mu \geq 0$.

Next, we introduced the following family of linear operators $I_{\mu,p}^{\lambda}(a, b;\ c): A_p \to A_p$; defined by

$$I_{\mu,p}^{\lambda}(a, b;\ c) f(z) = f_{\mu,p}^{\lambda}(a, b;\ c)(z) * f(z), \quad (1.5)$$

where $\lambda > -p$; $\mu \geq 0; z \in U$,

where$f_{\mu,p}^{\lambda}(a, b;\ c)(z)$is the function defined in terms of the Hadmard product as follows: For $z \in U$; $\mu \geq 0$,

$$f_{\mu,p}(a, b;\ c)(z) * f_{\mu,p}^{\lambda}(a, b;\ c)(z) = \frac{z^p}{(1-z)^{\lambda+p}}, \quad (1.6)$$

where$f_{\mu,p}(a, b;\ c)(z)$is given by(1.4).

We also note that the operator $I_{\mu,p}^{\lambda}(a, b;\ c) f(z)$generalizes several previously studied familiar operator, and we will show some of interesting particular cases as follows -

(i) I $I_{\mu,1}^{\lambda}(a, b;\ c) f(z) = I_{\mu}^{\lambda}(a, b;\ c) f(z)$, where $I_{\mu}^{\lambda}(a, b;\ c) f(z)$is the Srivastava-Khairnar-More operator [14]

(ii)$I_{0,1}^{\lambda}(a, b;\ c) f(z) = I^{\lambda}(a, b;\ c) f(z)$,where the operator $I^{\lambda}(a, b;\ c) f(z)$was introduced by Noor [12]

(iii)$I_{0,p}^{\lambda}(a, 1;\ c) f(z) = I_p^{\lambda}(a, c) f(z)$,where$I_p^{\lambda}(a, c)$ is the Cho-Kwon-Srivastava operator [2]

(iv)$I_{0,p}^{n}(a, n+1;\ c) f(z) = I^n$, where$I^n$ is the Noor integral operator [11]

(v)$I_{\mu,p}^{\lambda}(a, b;\ c) f(z)$was introduced by Huo Tang et. al. [5]

Since

$$\frac{z^p}{(1-z)^{\lambda+p}} = \sum_{k=1}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k+p}, \lambda > -p; z \in U, \quad (1.7)$$

By using (1.2), (1.4) and (1.7) in (1.6), we get

$$\left(\sum_{k=1}^{\infty} \frac{\big(1 + \mu(k+p-1)\big)(a)_k (b)_k}{(c)_k} \frac{z^{k+p}}{k!} \right) * f_{\mu,p}^{\lambda}(a, b;\ c)(z) = \sum_{k=1}^{\infty} (\lambda+p)_k \frac{z^{k+p}}{k!}$$

Therefore, the function $f_{\mu,p}^{\lambda}(a, b;\ c)(z)$has the following explicit form -

$$f_{\mu,p}^{\lambda}(a, b;\ c)(z) = \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{\big(1 + \mu(k+p-1)\big)(a)_k (b)_k} a_{k+p} z^{k+p},\ z \in U, (1.8)$$

For the complex parameters $\alpha_1, \dots \alpha_q$ and$\beta_1, \dots \beta_s$ where$\big(\beta_j \notin \bar{z}_0 = \{0, -1, -2, \dots s\}; j = 1,2, \dots s\big)$, we consider the generalized hypergeometric function $qF_s\big(\alpha_1, \dots \alpha_q;\ \beta_1, \dots \beta_s; z\big)$ by

$$qF_s\big(\alpha_1, \dots \alpha_q;\ \beta_1, \dots \beta_s; z\big) = \sum_{k=1}^{\infty} \frac{(\alpha_1)_{k,} \dots (\alpha_q)_{k,}}{(\beta_1)_{k,} \dots (\beta_s)_{k,} (1)_k} z^k, (1.9)$$

where$(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup 0;\ z \in U)$and $(x)_n$ is the Pochhammer symbol defined by (1.3).

Let

$$h_{p,q,s}(\alpha_1, \beta_1; z) = z^p . qF_s\big(\alpha_1, \dots \alpha_q;\ \beta_1, \dots \beta_s; z\big), \quad (1.10)$$

Corresponding to the function $h_{p,q,s}(\alpha_1, \beta_1; z)$and $f_{\mu,p}^{\lambda}(a, b;\ c)(z)$, and using the Hadmard product for $f(z) \in A_p, z \in U$, we get

$$I_{p,q,s,\mu}^{\lambda}(a, b, c, \alpha_1, \beta_1) f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\alpha_1)_{k,} \dots (\alpha_q)_{k,}}{(\beta_1)_{k,} \dots (\beta_s)_{k,} (1)_k} \frac{(\lambda+p)_k (c)_k}{\big(1 + \mu(k+p-1)\big)(a)_k (b)_k} a_{k+p} z^{k+p}, \quad (1.11)$$

and

$$z\left(I_{p,q,s,\mu}^{\lambda}(a, b, c, \alpha_1, \beta_1) f(z)\right)' = (\lambda+p) I_{p,q,s,\mu}^{\lambda+1}(a, b, c, \alpha_1, \beta_1) f(z) - \lambda I_{p,q,s,\mu}^{\lambda}(a, b, c, \alpha_1, \beta_1) f(z) \quad (1.12)$$

Let P denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_n z^n,$$

which are analytic and convex in U and satisfy the condition

$$Re\big(p(z)\big) > 0, z \in U.$$

By making use of the multiplier operator $I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)f(z)$ and the above mentionedprinciple of subordination, we define and investigate the following subclass of p-valent functions.

Definition 1.1 A function $f(z)\in A_p$ is said to be in the class $S_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1;\eta;h)$if it satisfies the differential subordination

$$\frac{1}{p-\eta}\left(\frac{z\left(I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)\right)'}{I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)}-\eta\right)\prec h(z), \qquad (1.13)$$

$z\in U; 0\le\eta<p; h\in P$ The purpose of this investigation is to establish results as inclusion relation, subordinationproperties, integral preserving property and argument estimate for the class$S_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1;\eta;h)$ by making use of the subordination relations included in the Lemmas of the following section.

2. Preliminary Results

The following Lemmas are needed to prove our main results.

Lemma 2.1 (See [4]) Let $\varsigma,\vartheta\in C$ and suppose that $p(z)$ is convex and univalent in U with $p(0)=1$ and $Re(\varsigma\, p(z)+\vartheta)>0$; $z\in U$. If $q(z)$ is analytic in U with $p(0)=1$, then the subordination

$$q(z)+\frac{zq'(z)}{\varsigma\, p(z)+\vartheta}\prec p(z) implies\ q(z)\prec p(z), z\in U.$$

Lemma 2.2 (See [9]) Let$h(z)$ be convex and univalent in U and $g(z)$ be analytic in U with $Re(g(z))\ge 0$; $z\in U$. If $\varphi(z)$ is analytic in U with $\varphi(0)=h(0)$, then the subordination

$$\varphi(z)+g(z)\varphi'(z)\prec h(z)\ implies\ that\ \varphi(z)\prec h(z), z\in U.$$

Lemma 2.3 (See [3]) Let q be analytic in U with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$. Ifthere exists two points $z_1,z_2\in U$ such that

$$-\frac{\pi}{2}\psi_1=arg(q(z_1))<arg(q(z))<arg(q(z_2))=\frac{\pi}{2}\psi_2$$

for some ψ_1 and ψ_2 $(\psi_1,\psi_2)>0$ and for all z $(|z|<|z_1|=|z_2|)$, then

$$\frac{z_1q'(z_1)}{q(z_1)}=-i\left(\frac{\psi_1+\psi_2}{2}\right)l\ and\ \frac{z_2q'(z_2)}{q(z_2)}=-i\left(\frac{\psi_1+\psi_2}{2}\right)l$$

where

$$l\ge\frac{1-|b|}{1+|b|}\ and\ b=itan\frac{\pi}{4}\left(\frac{\psi_1-\psi_2}{\psi_1+\psi_2}\right)$$

Lemma 2.4 (See [13]) The function

$$(1-z)^{\gamma}\equiv exp\big(\gamma log(1-z)\big), (\gamma\neq 0)$$

is univalent if and only if γ is either in the closed disk $|\gamma-1|\le 1$ or in the closed disk$|\gamma+1|\le 1$.

Lemma 2.5 (See [10]) Let $q(z)$ be univalent in U and $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $q(U)$ with $\varphi(w)\neq 0$when $w\in q(U)$.Set

$$Q(z)=zq'(z)\varphi\big(q(z)\big), h(z)=\theta\big(q(z)\big)+Q(z)$$

and suppose that

1. Q(z) is starlike (univalent) in U;
2. $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\theta'(q(z))}{\varphi(q(z))}+\frac{zQ'(z)}{Q(z)}\right)>0, (z\in U)$.

If p(z) is analytic in U with $p(0)=q(0)$ and $p(U)\subset D$, and

$$\theta'(p(z))+zp'(z)\varphi\big(p(z)\big)\prec\theta\big(q(z)\big)+z\big(q'(z)\big)\varphi\big(q(z)\big)=h(z),$$

then$p(z)\prec q(z)$, and q is the best dominant.

3. Main Results

We begin our main results by proving the following theorem in which inclusion relationship for the class $S_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1;\eta;h)$is obtained.

Theorem 3.1 Let class $f\in S_{p,q,s,\mu}^{\lambda+1}(a,b,c,\alpha_1,\beta_1;\eta;h)$with$Re((p-\eta)h(z)+\lambda+\eta)>0$.Then

$$S_{p,q,s,\mu}^{\lambda+1}(a,b,c,\alpha_1,\beta_1;\eta;h)\subset S_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1;\eta;h)$$

Proof: Let $f\in S_{p,q,s,\mu}^{\lambda+1}(a,b,c,\alpha_1,\beta_1;\eta;h)$ and assume that

$$q(z)=\frac{1}{p-\eta}\left(\frac{z\big(I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)f(z)\big)'}{I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)f(z)}-\eta\right) \qquad (3.1)$$

Then $q(z)$ is analytic in $U$with $q(0)=1$. Combining (1.12)and (3.1), we obtain

Now, by logarithmic differentiating on both sides of (3.2) and using (3.1), we get

$$(p+1)\frac{I_{p,q,s,\mu}^{\lambda+1}(a,b,c,\alpha_1,\beta_1)f(z)}{I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)f(z)}=(p-\eta)q(z)+\lambda+\eta \qquad (3.2)$$

Now, by logarithmic differentiating on both sides of (3.2) and using (3.1), we get

$$\frac{1}{p-\eta}\left(\frac{z\big(I_{p,q,s,\mu}^{\lambda+1}(a,b,c,\alpha_1,\beta_1)f(z)\big)'}{I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)f(z)}-\eta\right)=q(z)+\frac{zq'(z)}{(p-\eta)q(z)+\lambda+\eta}\prec h(z). \qquad (3.3)$$

Since

$$Re\big((p-\eta)q(z)+\lambda+\eta\big)>0$$

an application of Lemma 2.1 to (3.3) yields

$$q(z)\prec h(z), z\in U;$$

which implies that $f\in S_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1;\eta;h)$.

Theorem 3.2 Let $1<\sigma<2$ and $\gamma\neq 0$ be a real number satisfying either

$$|2\gamma(\sigma-1)(p+\lambda)-1|\le 1$$

or

$$|2\gamma(\sigma-1)(p+\lambda)+1|\le 1$$

If $f\in A_p$ satisfies the condition

$$Re\left(1+\frac{I_{p,q,s,\mu}^{\lambda+1}(a,b,c,\alpha_1,\beta_1)f(z)}{I_{p,q,s,\mu}^{\lambda}(a,b,c,\alpha_1,\beta_1)f(z)}-\eta\right)>2-\sigma+\frac{\lambda-1}{\lambda+p}, \quad z\in U. \qquad (3.4)$$

Then

$$\left(zI^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)^{\gamma} \prec q(z_1)=\frac{1}{(1-z)^{2\gamma(\sigma-1)(p+\lambda)}},$$

where $q_1(z)$ is the best dominant.

Proof: Suppose that

$$p(z):=\left(zI^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)^{\gamma}.$$

Hence by taking logarithmic differentiation and using (1.12), we get

$$\frac{zp'(z)}{p(z)}=\gamma(1-\lambda)+\gamma(p+\lambda)\frac{zI^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)}{I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)} \quad (3.5)$$

Combining (3.4) and (3.5), we find that

$$1+\frac{zp'(z)}{\gamma(p+\lambda)p(z)} \prec \frac{1+(2\sigma-3)z}{1-z}.$$

If we choose

$$\theta(w)=1, q_1(z)=\frac{1}{(1-z)^{2\gamma(\sigma-1)(p+\lambda)}} \text{ and } \varphi(w)=\frac{1}{\gamma w(p+\lambda)},$$

then by assumption of the theorem and making use of Lemma 2.4, we known that $q_1(z)$ is univalent in U. It follows that

$$Q(z)=zq'(z)\varphi(q(z))=\frac{2(\sigma-1)z}{1-z}$$

and

$$\theta(q(z))+Q(z)=\frac{1+(2\sigma-3)z}{1-z}=\text{h}(z).$$

If the domain D is defined by

$$q(U)=\left\{w:\left|w^{\frac{1}{\eta}}-1\right|<\left|w^{\frac{1}{\eta}}\right|,\eta=2\gamma(\sigma-1)(p+\lambda)\right\}\subset D,$$

then it is easily verified that the conditions of Lemma 2.5 are satisfied. Hence $q_1(z)\prec q(z)$.

Theorem 3.3 Let $f\in S^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1;\eta;h)$ with

$$Re\big((p-\eta)h(z)+\lambda+\eta\big)>0, z\in U.$$

Then the integral operator F defined by

$$F(z)=\frac{\lambda+p}{z^{\lambda}}\int_0^z t^{\lambda-1}f(t)dt, z\in U$$

belongs to the class $S^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1;\eta;h)$.

Proof: Let $f\in S^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1;\eta;h)$. Then, from (3.6), it can be easily verified that

$$z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)F(z)\right)'+\lambda I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)F(z) = (\lambda+p)I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)$$

By setting

$$q_2(z):=\frac{1}{p-\eta}\left(\frac{z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)F(z)\right)'}{I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)F(z)}-\eta\right).$$

We observe that $q_2(z)$ is analytic in U with $q_2(0)=0$. It follows from (3.7) and (3.8) that

$$(p+\lambda)\frac{zI^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)}{I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)F(z)}=\lambda+\eta+(p-\eta)q_2(z). \quad (3.9)$$

By logarithmic differentiation of (3.9) and making use of (3.8), it follows that

$$q_2(z)+\frac{zq_2'(z)}{\lambda+\eta+(p-\eta)q_2(z)} =\frac{1}{p-\eta}\left(\frac{z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)'}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)}-\eta\right) \prec h(z),$$

Since

$$Re\big(\lambda+\eta+(p-\eta)h(z)\big)>0, z\in U,$$

Then by Lemma 2.1, we get

$$q_2(z)=\frac{1}{p-\eta}\left(\frac{z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)F(z)\right)'}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)F(z)}-\eta\right)\prec h(z).$$

Which Implies the $F(z)\in S^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1;\eta;h)$.

Theorem 3.4 Let $f\in A_p$, $0<\tau_1,\tau_2\le 1$ and $0\le\eta<p$. If

$$-\frac{\pi}{2}\tau_1<arg\left(\frac{z\left(I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)'}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}-\eta\right)<\frac{\pi}{2}\tau_2$$

For some

$g\in S^{\lambda+1}_{p,q,s,\mu}\left(a,b,c,\alpha_1,\beta_1;\eta;\frac{1+Az}{1+Bz}\right), -1\le B<A\le 1$, then

$$-\frac{\pi}{2}\psi_1<arg\left(\frac{z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)'}{I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}-\eta\right)<\frac{\pi}{2}\psi_2,$$

where ψ_1 and ψ_2 $(0<\psi_1,\psi_2\le 1)$ are the solutions of the following equations

$$\tau_1=\begin{cases}\psi_1+\frac{2}{\pi}tan^{-1}\left(\frac{(1-|b|)(\psi_1+\psi_2)cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda\right)+(1-|b|)(\psi_1+\psi_2)sin\frac{\pi}{2}t}\right) & (B\neq-1)\\ \psi_1 & (B=-1)\end{cases}$$

and

$$\tau_2=\begin{cases}\psi_2+\frac{2}{\pi}tan^{-1}\left(\frac{(1-|b|)(\psi_1+\psi_2)cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda\right)+(1-|b|)(\psi_1+\psi_2)sin\frac{\pi}{2}t}\right) & (B\neq-1)\\ \psi_2 & (B=-1)\end{cases}$$

With $b=tan\frac{\pi}{2}\left(\frac{\psi_1-\psi_2}{\psi_1+\psi_2}\right)$ and $t=\frac{2}{\pi}sin^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+(\eta+\lambda)(1-B^2)}\right)$

Proof: Supposethat

$$q_3(z):=\frac{1}{p-\gamma}\left(\frac{z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)'}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}-\gamma\right), \quad (3.10)$$

With $0\le\gamma<p$ and $g\in S^{\lambda+1}_{p,q,s,\mu}\left(a,b,c,\alpha_1,\beta_1;\eta;\frac{1+Az}{1+Bz}\right), -1\le B<A\le 1$, then $q_3(z)$ is analytic in U with $q_3(0)=1$. It follows from (1.12) and (3.10) that

$$[(p-\gamma)q_3(z)+\gamma]I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z) =(p+\lambda)I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z) -\lambda I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z). \quad (3.11)$$

Differentiating both sides of (3.11) and multiplying resulting equation by z, we get

$(p-\gamma)zq_3'(z)I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)$

$$+[(p-\gamma)q_3(z)+\gamma]z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)\right)'$$
$$=(p+\lambda)\left(zI^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)'$$
$$-\lambda z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)'.$$

Since $g\in S^{\lambda+1}_{p,q,s,\mu}\left(a,b,c,\alpha_1,\beta_1;\eta;\frac{1+Az}{1+Bz}\right)$,by Theorem 3.1, we know that $g\in S^{\lambda}_{p,q,s,\mu}\left(a,b,c,\alpha_1,\beta_1;\eta;\frac{1+Az}{1+Bz}\right)$. Let

$$q_4(z):=\frac{1}{p-\eta}\left(\frac{z\left(I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)\right)'}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}-\eta\right), \tag{3.12}$$

Combining (1.12)(with f replace by g) and (3.12), we easily get

$$\frac{I^{\lambda}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}=\frac{p+\lambda}{(p-\eta)q_4(z)+\eta+\lambda}. \tag{3.13}$$

Now, from (3.9), (3.12) and (3.13), we find that

$$\frac{1}{p-\gamma}\left(\frac{z\left(I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)\right)'}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}-\gamma\right)$$
$$=q_3(z)+\frac{zq_3{}'(z)}{(p-\eta)q_4(z)+\eta+\lambda} \tag{3.14}$$

Since

$$q_4(z)\prec\frac{1+Az}{1+Bz}\quad(-1\le B<A\le 1),$$

it is easy to see that

$$\left|q_4(z)-\frac{1-AB}{1-B^2}\right|<\frac{A-B}{1-B^2}(z\in U;B\neq -1) \tag{3.15}$$

and

$$Re\big(q_4(z)\big)>\frac{1-A}{2},\qquad (z\in U;B=-1) \tag{3.16}$$

We now easily find from (3.15) and (3.16) that

$$\left|(p-\eta)q_4(z)+\eta+\lambda-\frac{(\eta+\lambda)(1-B^2)+(p-\eta)(1-AB)}{1-B^2}\right|<\frac{(p-\eta)(A-B)}{1-B^2},\quad (B\neq -1)$$

And

$$Re\big((p-\eta)q_4(z)+\eta+\lambda\big)>\frac{(p-\eta)(1-A)}{2}+\eta+\lambda,\quad (B=-1)$$

If we set

$$(p-\eta)q_4(z)+\eta+\lambda=r\exp\left(i\frac{\pi}{2}\theta\right),$$

Where

$$-\rho<\theta<\rho;\ \rho=\frac{(p-\eta)(A-B)}{(\eta+\lambda)(1-B^2)+(p-\eta)(1-AB)},\quad (B\neq -1),$$

And

$$-1<\theta<1,\qquad (B=-1)$$

then

$$\frac{(p-\eta)(1-A)}{1-B}+\eta+\lambda<r<\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda,\quad (B\neq -1)$$

$$\frac{(p-\eta)(1-A)}{2}+\eta+\lambda<r,\qquad (B=-1).$$

Since $q_3(z)$ is analytic in $U$with $q_3(0)=1$, an application of Lemma 2.2 to (3.14) yields $q_3(z)\prec h(z)$. Now, assume that

$$Q(z)=\frac{1}{p-\gamma}\left(\frac{z\left(I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)f(z)\right)'}{I^{\lambda+1}_{p,q,s,\mu}(a,b,c,\alpha_1,\beta_1)g(z)}-\gamma\right),\quad (0\le\gamma<p) \tag{3.17}$$

Then by means of (3.14) and (3.17) we have

$$argQ(z)=arg\big(q_3(z)\big)+arg\left(1+\frac{zq_3{}'(z)}{[(p-\eta)q_4(z)+\eta+\lambda]q_3(z)}\right)$$

If there exist two points $z_1,z_2\in U$ such that

$$-\frac{\pi}{2}\psi_1=arg\big(q_3(z_1)\big)<arg\big(q_3(z)\big)<arg\big(q_3(z_2)\big)=\frac{\pi}{2}\psi_2,$$

by Lemma 2.3, we know that

$$\frac{z_1q_3{}'(z_1)}{q_3(z_1)}=-i\left(\frac{\psi_1+\psi_2}{2}\right)l\ and\ \frac{z_2q_3{}'(z_2)}{q_3(z_2)}=i\left(\frac{\psi_1+\psi_2}{2}\right)l$$

Where

$$i\ge\frac{1-|b|}{1+|b|}\ and\ b=itan\frac{\pi}{4}\left(\frac{\psi_1-\psi_2}{\psi_1+\psi_2}\right)$$

We have the following two cases:

1. When $B\neq -1$, we have

$$\big(argQ(z_1)\big)=arg\big(q_3(z_1)\big)+arg\left(1+\frac{z_1q_3{}'(z_1)}{[(p-\eta)q_4(z_1)+\eta+\lambda]q_3(z_1)}\right)$$
$$=-\frac{\pi}{2}\psi_1+arg\left(1-il\left(\frac{\psi_1+\psi_2}{2}\right)r^{-1}e^{-i\frac{\pi}{2}\theta}\right)$$
$$=-\frac{\pi}{2}\psi_1+arg\left(1-\frac{l}{2r}(\psi_1+\psi_2)cos\frac{\pi}{2}(1-\theta)+\frac{il}{2r}(\psi_1+\psi_2)sin\frac{\pi}{2}(1-\theta)\right)$$
$$\le-\frac{\pi}{2}\psi_1-tan^{-1}\left(\frac{l(\psi_1+\psi_2)sin\frac{\pi}{2}(1-\theta)}{2r+l(\psi_1+\psi_2)cos\frac{\pi}{2}(1-\theta)}\right)$$
$$\le-\frac{\pi}{2}\psi_1-tan^{-1}\left(\frac{(1-|b|)(\psi_1+\psi_2)cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda\right)+(1-|b|)(\psi_1+\psi_2)sin\frac{\pi}{2}t}\right)$$
$$=-\frac{\pi}{2}\tau_1$$

and

$$arg\big(Q(z_2)\big)=arg\big(q_3(z_2)\big)+arg\left(1+\frac{z_2q_3{}'(z_2)}{[(p-\eta)q_4(z_2)+\eta+\lambda]q_3(z_2)}\right)$$
$$\ge\frac{\pi}{2}\psi_2+tan^{-1}\left(\frac{(1-|b|)(\psi_1+\psi_2)cos\frac{\pi}{2}t}{2(1+|b|)\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda\right)+(1-|b|)(\psi_1+\psi_2)sin\frac{\pi}{2}t}\right)$$
$$=\frac{\pi}{2}\tau_2.$$

2. For the case $B = 1$, we similarly obtain

$$\left(argQ(z_1)\right) = arg\left(q_3(z_1) + \frac{z_1 q_3{}'(z_1)}{[(p-\eta)q_4(z_1)+\eta+\lambda]q_3(z_1)}\right) \le -\frac{\pi}{2}\psi_1,$$

and

$$arg\left(Q(z_2)\right) = arg\left(q_3(z_2) + \frac{z_2 q_3{}'(z_2)}{[(p-\eta)q_4(z_2)+\eta+\lambda]q_3(z_2)}\right) \ge \frac{\pi}{2}\psi_2.$$

The above two cases contradict the assumptions of the Theorem 3.4. Hence the proof is complete.

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